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Generalized octahedra and cliques in intersection graphs of uniform hypergraphs[1]

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Abstract

It is shown that k-uniform hypergraphs with m edges contain at most $O(m^{\binom{2k}{k}})$ maximal sets of pairwise intersecting hyperedges, and ℓ-intersection graphs $G=(V,E)$ of k-uniform hypergraphs contain $O(|V|^{\binom{2(k-\ell+1)}{k-\ell+1}})$ maximal cliques. In case $\ell=k-2$, the result is improved to $O(|V||E|)$. For every fixed k, the results imply polynomial-time algorithms for computing maximum sets of pairwise intersecting hyperedges in k-uniform hypergraphs, respectively maximal cliques in their intersection graphs. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

All graphs and hypergraphs considered are finite. *Intersection cliques* in hypergraphs are maximal sets of pairwise intersecting hyperedges. This concept may be more natural in an intersection graph setting, since intersection cliques correspond just to the cliques of the intersection graph of the hypergraph. By *cliques*, we always mean maximal complete subgraphs.

Intersection cliques in 2-uniform hypergraphs are rather easy to describe — either they consist of three hyperedges forming a triangle or all hyperedges of the subfamily have some common vertex (the so-called 'Helly property'). For 3-uniform hypergraphs, a description of all intersection cliques is given in Section 4 of the present paper, but a complete description of the possible intersection cliques in k-uniform hypergraphs for higher k seems to be difficult.

Nevertheless it turns out that for every fixed k, a maximum intersecting family in a k-uniform hypergraph H can be found in time polynomial in the vertex and hyperedge number of H. Note that this parameter is both useful for hypergraphs, since it is a

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lower bound of the edge chromatic number (or chromatic index) of the hypergraph, as for intersection graphs, where it equals the clique number, the size of a maximum clique (cf. Section 5).

The result follows from the fact that the number of all intersection cliques in every k-uniform hypergraph H is bounded by some polynomial in vertex and hyperedge number of H (Corollary 2). According to results in [2,11], all that has to be decided is the question of whether some generalized octahedron $\overline{cK_2}$ is not the intersection graph of some k-uniform hypergraph, for fixed k. But this question can be answered by a result of Bollobas, cf. [4, p. 66], as will be shown in Section 2.

Using a generalization by Füredi [5], similar results will also be given for ℓ-*intersection cliques* — maximal sets of hyperedges where every two members have at least ℓ common vertices. Actually the motivation for the present work was the observation that $(k-1)$-intersection graphs of k-uniform hypergraphs do not have more cliques than edges [8].

Although both tools, that in [2,11], as well as that in [5,4] are sharp, their combination is not. Far better bounds on the number of cliques in $\Gamma_{k,k-2}$ can be found by a somewhat tedious examination of all types of cliques occurring. This will be done in Section 4. Section 3 presents some lower bounds for the number of cliques in certain graphs of $\Gamma_{k,\ell}$.

1.1. Notations and prerequisites

A hypergraph $\{S_v \mid v \in V\}$ is k-*uniform* if all sets S_i have equal size k. Throughout this paper we only consider uniform hypergraphs. This is not really a restriction, since by enlarging the hyperedges of any general hypergraph H, we always arrive at some $r(H)$-uniform hypergraph with the same intersection pattern, where $r(H)$ denotes the rank of H.

Recall that the *intersection graph* (or *line graph*) of a simple hypergraph $\{S_v \mid v \in V\}$ has the index set V as vertex set, and the two vertices $v \neq w$ are adjacent if S_v and S_w have nonempty intersection. The ℓ-intersection graph of $\{S_v \mid v \in V\}$, has again V as vertex set, and $v \neq w \in V$ are adjacent if $|S_v \cap S_w| \geqslant \ell$. Let for $1 \leqslant \ell < k$, $\Gamma_{k,\ell}$ denote the set of all ℓ-intersection graphs of simple k-uniform hypergraphs. In the case of ordinary intersection, instead of $\Gamma_{k,1}$ we often simply write Γ_k.

Note that $\Gamma_{k,\ell} \subseteq \Gamma_{k+1,\ell}$ and $\Gamma_{k,\ell} \subseteq \Gamma_{k+1,\ell+1}$. We just have to enlarge the hyperedges by adding points, either one new private point to every hyperedge, or one new common point to all hyperedges.

2. Generalized octahedra in Γ_k

Theorem 1. *For every* $k \geqslant 2$,

$$\overline{\frac{1}{2}\binom{2k-2\ell+2}{k-\ell+1}K_2} \in \Gamma_{k,\ell}, \quad \textit{but} \quad \overline{\left(\frac{1}{2}\binom{2k-2\ell+2}{k-\ell+1}+1\right)K_2} \notin \Gamma_{k,\ell}.$$

Proof. (1) The $(k-\ell+1)$-element subsets of the set $\{1,2,\dots,2k-2\ell+2\}$ intersect each other with the exception of pairs of complementary sets. If we insert the $\ell-1$ elements $2k-2\ell+3,\dots,2k-\ell+1$ into each one of these sets, we get a k-uniform hypergraph whose ℓ-intersection graph equals

$$\overline{\frac{1}{2}\binom{2k-2\ell+2}{k-\ell+1}K_2}.$$

(2) That we cannot achieve more follows from a result in [5]. Assume we have a representation of $\overline{nK_2}$ with distinct sets $A_1,A_2,\dots,A_n,B_1,B_2,\dots,B_n$. We define $U_i:=V_{n+i}:=A_i$ and $U_{n+i}:=V_i:=B_i$ for $1\leqslant i\leqslant n$ to obtain k-element sets $U_1,U_2,\dots,U_{2n}$, $V_1,V_2,\dots,V_{2n}$ such that $|U_j\cap V_h|>\ell-1$ if and only if $j=h$. Füredi's result implies

$$2n\leqslant\binom{2k-2(\ell-1)}{k-(\ell-1)}. \qquad \square$$

Note that the classes $\Gamma_{k,\ell}$ are closed under induced subgraphs. It has been shown in [2,11] that graphs $G=(V,E)$ without induced subgraphs isomorphic to $\overline{pK_2}$ contain at most $|V|^{2(p-1)}$ cliques.

Corollary 2. *Every graph $G=(V,E)$ in $\Gamma_{k,\ell}$ has at most*

$$\mathrm{O}(|V|^{\binom{2k-2\ell+2}{k-\ell+1}})$$

cliques. Every k-uniform hypergraph with m hyperedges has at most

$$\mathrm{O}(m^{\binom{2k-2\ell+2}{k-\ell+1}})$$

ℓ-intersection cliques.

3. Lower bounds for the number of cliques

Recall that a *finite projective plane* of order n is a hypergraph H where every hyperedge contains at least 2 points, every two points lie in exactly one hyperedge (which implies linearity of the hypergraph), every two hyperedges have nonempty intersection, and there are four points such that no three of them are contained in a hyperedge [3]. It is well-known that H must be $(n+1)$-uniform with n^2+n+1 points and hyperedges. Note that the intersection graph is complete, therefore, finite projective planes are representations of complete graphs. Even more is true:

Lemma 3. *Every projective plane of order $k-1$ in a k-uniform hypergraph H is an intersection clique.*

Proof. Assume that the projective plane $\mathscr{P}=\{S_1,\dots,S_{k^2-k+1}\}$ is not an intersection clique of H, and let T be another hyperedge intersecting every S_i with $1\leqslant i\leqslant k^2-k+1$.

Let $S := \bigcup_{i=1}^{k^2-k+1} S_i$. Choose any point $a \in T \cap S$. The hyperedge T intersects exactly k of the S_i in a, say $S_1, \dots S_k$.

Since $S = \bigcup_{i=1}^{k} S_i$, w.l.g. $|T \cap S_k| \geqslant 2$. Since $T \neq S_k$, we can find $b \in S_k \backslash T$. Let w.l.g. $S_k, S_{k+1}, \dots, S_{2k-1}$ be the members of $\mathscr{P}$ containing b. The subsets $S_i \backslash \{b\}$ and $S_j \backslash \{b\}$ of $S \backslash S_k$ are disjoint for $k+1 \leqslant i < j \leqslant 2k-1$ — this implies that at least $k-1$ further points of T are necessary to meet every one of $S_{k+1}, \dots, S_{2k-1}$. In addition at least two points of $T \cap S_k$, T would have to contain at least $k+1$ points, a contradiction. □

Theorem 4. *If $k-1$ is a prime power, then there are arbitrarily large graphs $G = (V, E) \in \Gamma_k$ with $\Omega(|V|^{(k^2-k+1)/k})$ cliques.*

Proof. Consider the intersection graph of some family $\binom{M}{k}$. It has $\binom{|M|}{k} = \Theta(|M|^k)$ vertices and $\frac{1}{2}\binom{|M|}{k}(\binom{|M|}{k} - \binom{|M|-k}{k} - 1) = \Theta(|M|^{2k-1})$ edges. In every subset of order k^2-k+1 of M there is some projective plane of order $k-1$ by Veblen and Bussey's Theorem (see [3]) on the existence of finite projective planes of prime power order. By Lemma 3, each of them is the representation of some clique in G. All these cliques are distinct, so there are at least $\binom{|M|}{k^2-k+1} = \Omega(|V|^{(k^2-k+1)/k})$ cliques in G. □

For $k=3$ these graphs in Γ_3 have $\Omega(|V|^{7/3})$ and also $\Omega(|V|^{2/3}|E|)$ cliques.

These bounds give also lower bounds for the possible number of cliques of graphs in $\Gamma_{k,\ell}$. Since $\Gamma_{k-\ell+1,1} \subseteq \Gamma_{k,\ell}$, there are arbitrarily large graphs $G = (V, E) \in \Gamma_{k,\ell}$ with

$$\Omega\left(|V|^{\frac{k^2+\ell^2-2k\ell+k-\ell+1}{k-\ell+1}}\right)$$

cliques, provided $k-\ell$ is a prime power.

4. Cliques in the graphs in $\Gamma_{k,k-2}$

The gap between the upper bounds for the number of cliques, derived from the nonrepresentability of certain generalized octahedra in Section 2, and the lower bounds in Section 3 is enormous. In this section we will derive a much better upper bound in the case $\ell = k-2$, which almost agrees with the lower bound. This result may indicate that the lower bounds are sharper than the upper bounds.

Very helpful in the investigation of line graphs, or even more general, $(k-1)$-intersection graphs of k-uniform hypergraphs (see [6,8]) was classifying the cliques. If we look at the representatives $S_1, S_2, \dots, S_t$ of the vertices of a clique in a member $G = (V, E)$ of $\Gamma_{k,k-1}$ (i.e. on a $k-1$-intersection clique in a k-uniform hypergraph H), then there are only two types. One type, where the union of all corresponding k-sets contains $k+1$ points — then the clique has at most $k+1$ vertices too. Cliques of this type share at most one vertex, therefore there are at most $|E|$ cliques of this type. In the other type, the corresponding sets share $k-1$ points. Then there is no bound on

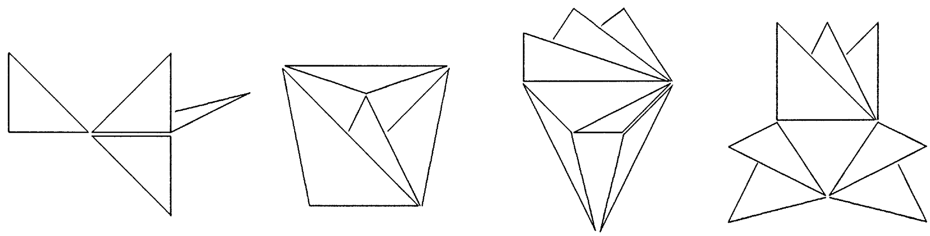

Fig. 1. Examples of intersection cliques in 3-uniform hypergraphs.

the number of points in the representation involved, or on the number of vertices of the clique. Nevertheless, the clique has some bounded description in the representation: All we have to give are the $k-1$ common points. These points are uniquely described by just giving two adjacent vertices of the clique. Overall, G has at most $2|E|$ cliques. Actually

Theorem 5 (Le and Prisner [8]). *Every graph* $G=(V,E)\in\Gamma_{k,k-1}$ *has at most* $|E|$ *cliques.*

The result is sharp up to a constant, since $L(K_n)$ has $3\binom{n}{3}$ edges and $\binom{n}{3}+n$ cliques.

For $(k-2)$-intersection graphs of k-uniform hypergraphs there is a variety of shapes for the representations of cliques, see Fig. 1 for a few examples in the case $k=3$ and $\ell=1$. However, it turns out that we can classify them by certain types, all having some description involving at most $k+4$ points of the representation. Moreover, there is a constant $C(k)$ such that every triangle of G lies in at most $C(k)$ cliques, as we shall see below. This gives a tight bound for the number of cliques.

Let $G=(V,E)$ be the $(k-2)$-intersection graph of the k-uniform hypergraph H. The most natural type for a $(k-2)$-intersection clique $\mathscr{C}$ is when $|\bigcap_{S\in\mathscr{C}} S|=k-2$. For $k=3$, this is the so-called *Helly-property*. Such representations of cliques are called *type A*. The leftmost clique in Fig. 1 is an example.

In the following we assume that $\mathscr{C}$ is not of type A. We call a $(k-1)$-element subset M of $\bigcup_{S\in\mathscr{C}} S$ a *Helly set* (with respect to $\mathscr{C}$) if $|M\cap S|\geqslant k-2$ for every $S\in\mathscr{C}$.

The property to have at least $k-2$ common members with every Helly set is necessary but not sufficient for an edge of H to lie in $\mathscr{C}$. However, every edge of H that includes some Helly set must belong to $\mathscr{C}$. Therefore, a description of $\mathscr{C}$ is complete if it contains all Helly sets, together with all those members of $\mathscr{C}$ that do not include any Helly set (with respect to $\mathscr{C}$). We call those $\mathscr{C}$-members *relevant*.

The *head* of $\mathscr{C}$ is the union of Helly sets and relevant $\mathscr{C}$-members.

Let us illustrate these notions in Fig. 1. The second intersection clique has six Helly sets, which form a $K_{2,3}$ when viewed as edges of some graph. Only one upper triangle is relevant. The four Helly sets in the third intersection clique form a $K_{1,3}$ with one edge subdivided; only the left triangle is relevant. In the fourth intersection clique, we have three Helly sets forming a triangle, and no relevant $\mathscr{C}$-set at all.

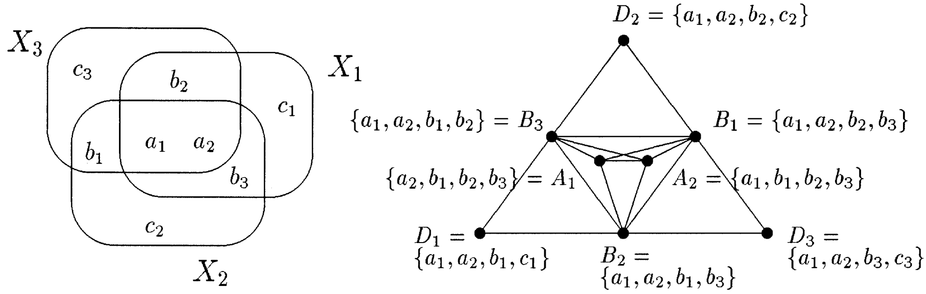

Fig. 2.

Lemma 6. *Every non-type-A $(k-2)$-intersection clique $\mathscr{C}$ of a k-uniform hypergraph H has three $\mathscr{C}$-members whose union contains the head.*

Proof. *Case* 1: $|S \cap T| = k-1$ for all $S, T \in \mathscr{C}$. Choose any two members $X = \{a_1, \ldots, a_k\}$ and $Y = \{a_2, \ldots, a_{k+1}\}$ of $\mathscr{C}$. Since we do not have type A, for every $2 \leqslant i \leqslant k$ there must be some $U_i \in \mathscr{C}$ avoiding a_i. But since no two $\mathscr{C}$-members have only $k-2$ common elements, $U_i = \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}\}$. No other k-element set can join $k-2$ elements with every one of these sets, so $\mathscr{C} = \{X, Y, U_2, \ldots, U_{k+1}\}$.

Case 2: There are $X, Y \in \mathscr{C}$ such that $|X \cap Y| = k-2$. If $\bigcup_{S \in \mathscr{C}} S$ contains just $k+2$ or $k+3$ points we are done, so assume in what follows $|\bigcup_{S \in \mathscr{C}} S| \geqslant k+4$.

Subcase 2.1: There are $X_1, X_2, X_3 \in \mathscr{C}$ with $|X_1 \cap X_2 \cap X_3| = k-3$, and $|X_1 \cap X_2| = |X_1 \cap X_3| = |X_2 \cap X_3| = k-2$. Assume $X_1 = \{a_1, \ldots, a_{k-3}, c_1, b_2, b_3\}$, $X_2 = \{a_1, \ldots, a_{k-3}, c_2, b_1, b_3\}$, and $X_3 = \{a_1, \ldots, a_{k-3}, c_3, b_1, b_2\}$, as on the left of Fig. 2. We call $U \in \mathscr{C}$ *inner* $\mathscr{C}$-set if it is included in $X_1 \cup X_2 \cup X_3$ and *outer* $\mathscr{C}$-set otherwise.

Now assume that a set P obeys $|P \cap X_i| \geqslant k-2$ for every $i = 1, 2, 3$. This implies

$$3|P \cap \{a_1, \ldots, a_{k-3}\}| + 2|P \cap \{b_1, b_2, b_3\}| + |P \cap \{c_1, c_2, c_3\}| \geqslant 3(k-2).$$

Certainly we get $|P| \geqslant k-1$, and in case $|P| = k-1$, it must be one of the sets

$$A_i = \{a_1, \ldots, a_{k-3}, b_1, b_2, b_3\} \backslash \{a_i\}, \quad i = 1, \ldots, k-3,$$
$$B_j = \{a_1, \ldots, a_{k-3}, b_1, b_2, b_3\} \backslash \{b_j\}, \quad i = 1, \ldots, 3, \text{ or}$$
$$D_h = \{a_1, \ldots, a_{k-3}, b_h, c_h\}, \quad i = 1, \ldots, 3.$$

The $(k-2)$-intersection graph of these $k+3$ sets is called Ω, see Fig. 2 for an example for $k = 5$.

Therefore, every outer $\mathscr{C}$-set must have the form $P \cup \{f\}$, with P being one of the sets A_i, B_j, or D_h. These sets A_i, B_j, or D_h are also the only possible Helly sets.

If there are no relevant outer $\mathscr{C}$-sets, then the head is $\bigcup_{i=1}^{3} X_i$ or a subset thereof, and we get no problems.

So let $P_1 \cup \{f\}$ be a relevant outer $\mathscr{C}$-set, where P_1 is a vertex of Ω. Then P_1 is not a Helly set (with respect to $\mathscr{C}$), therefore $|P_1 \cap Y| < k-2$ for some $Y \in \mathscr{C}$. Since

$|(P_1\cup\{f\})\cap Y|\geqslant k-2$, Y must be an outer $\mathscr{C}$-set of the form $Y=P_2\cup\{f\}$. Then $|P_1\cap P_2|=k-3$, $Y=P_2\cup\{f\}$ is also relevant, and the situation is symmetric in P_1 and P_2. Since P_1 and P_2 are not adjacent in Ω, w.l.g. $P_1=D_1$.

Now assume there were another relevant outer $\mathscr{C}$-set $P_3\cup\{g\}$ with $g\neq f$ and P_3 a vertex of Ω. In the same way as above, we may assume w.l.g. $P_3=D_i$ for some $i\in\{1,2,3\}$. Since $|D_1\cap D_i|=k-3$ for $i=2,3$, and since $|(D_1\cup\{f\})\cap(D_i\cup\{g\})|\geqslant k-2$, we conclude $i=1$. But then $|(D_1\cup\{g\})\cap(P_2\cup\{f\})|=|D_1\cap P_2|=k-3$, a contradiction. Thus the three $\mathscr{C}$-sets $D_1\cup\{f\},X_2,X_3$ cover $\bigcup_{i=1}^{3}X_i\cup\{f\}$, which, as we have shown, itself covers the head.

Subcase 2.2: For all $X,Y\in\mathscr{C}$ obeying $|X\cap Y|=k-2$ every further member $Z\in\mathscr{C}$ includes $X\cap Y$ or is contained in $X\cup Y$. Since we do not have type A, there must be some $Z\in\mathscr{C}$ contained in $X\cup Y\backslash\{a\}$ for some $a\in X\cap Y$. Since $|\bigcup_{S\in\mathscr{C}}S|\geqslant k+4$, there must be some $U\in\mathscr{C}$ with $U\nsubseteq X\cup Y$.

If $|Z\cap(X\cap Y)|=k-4$, then $|Z\cap U|\leqslant k-3$, a contradiction. Therefore we may assume $|Z\cap(X\cap Y)|=k-3$. Let w.l.g. $(X\cap Z)\backslash Y=\{b,c\}$ and $(Y\cap Z)\backslash X=\{d\}$. The hyperedge U contains exactly one of b,c, or d. It cannot be b or c — otherwise Y,Z, and U would form a forbidden configuration as in subcase 2.1, thus it must be d. Every further $Z'\in\mathscr{C}$ included in $X\cup Y$ must obey $(Z'\cap Y)\backslash X=\{d\}$. Every further $U'\in\mathscr{C}$ not included in $X\cup Y$ must obey $U'\cap(X\cup Y)=(X\cap Y)\cup\{d\}$. By a similar counting argument as in subcase 2.1, every Helly set must be contained in $X\cup Y$, and it is rather easy to see that just $(X\cap Y)\cup\{d\}$ and all sets $W\cup\{b,d\}$ and $W\cup\{c,d\}$, with $(k-3)$-element subsets W of $X\cap Y$, are Helly sets. Therefore, all such sets U' not contained in $X\cup Y$ are non-relevant, and the head is contained in $X\cup Y$. $\square$

Theorem 7. *For every integer $k\geqslant 3$, every graph $G=(V,E)\in\Gamma_{k,k-2}$ has* $O(|V||E|)$ *cliques.*

Proof. Let G be the $(k-2)$-intersection graph of the family $\{S_x\,|\,x\in V\}$ of k-element sets, and let $S=\bigcup_{x\in V}S_x$. Again, for cliques C of G we denote the corresponding subfamily of the representation by $\mathscr{C}=\{S_v\,|\,v\in C\}$.

Note that for every type A clique $\mathscr{C}$ the set $\bigcap_{S\in\mathscr{C}}S$ contains $k-1$ or $k-2$ elements, and that all these sets are pairwise incomparable. Moreover, for every clique $\mathscr{C}$ of type A containing an edge xy we have $\bigcap_{S\in\mathscr{C}}S\subseteq X\cap Y$, where $X\cap Y$ contains also $k-1$ or $k-2$ elements. Therefore there are at most $\binom{k-1}{k-2}|E|=(k-1)|E|$ cliques of type A in G.

Let t denote the number of triangles in G. For every three pairwise adjacent vertices $x,y,z\in V$ there is only a constant number of non-type-A cliques whose head is contained in $S_x\cup S_y\cup S_z$. Since x,y,z forms a triangle in G, the number of these cliques is $O(t)$,

Hence, the number of cliques in G is $O(t)=O(|E||V|)$. $\square$

The constants hidden in this O-notation depend on k. I do not know whether or not they have some common upper bound.

5. Applications

The *F-intersection graph* $\mathrm{Int}_F(G)$ of a graph G is defined as the vertex-intersection graph of the set of all subgraphs isomorphic to F in G, [1]. The *k-edge graph* $\nabla_k(G)$ is the intersection graph of the set of all cliques of cardinality smaller than k, and all complete subgraphs with k vertices of G [10]. Surely $\mathrm{Int}_{K_k}(G)$ is an induced subgraph of $\nabla_k(G)$, and $\nabla_k(G)=\mathrm{Int}_{K_k}(G^\dagger)$ with $G^\dagger$ being obtained from G by 'blowing up' all cliques of cardinality smaller than k.

It has been shown in [7] that computing the independence number is $\mathcal{NP}$-complete even for graphs $\mathrm{Int}_F(G)$ if F is connected with at least 3 vertices. But at least the clique number of F-intersection graphs can be computed in polynomial time. This can be done by listing all cliques of the graph, which can be achieved by an algorithm in [12] in time $O(n^3c)$, where n and c denote number of vertices and cliques, respectively.

Corollary 8. *The clique number can be computed*

- *in time* $O(|V|^4|E|)$ *for* P_3*-intersection graphs,* K_3*-intersection graphs, or* 3*-edge graphs* $G=(V,E)$,
- *in time* $O(|V|^{4^k+3})$ *for k-edge graphs, and for F-intersection graphs if F has* $k\geqslant 4$ *vertices.*

The results of the paper show also that the maximum cardinality $\omega(H)$ of an intersecting family in the k-uniform hypergraph $H=\{S_i \mid i\in I\}$ can be computed in polynomial time $O(|I|^{4^k+3})$ (for fixed $k\geqslant 4$), respectively $O(|I|^6)$ for $k=3$. This gives a potentially better, polynomially-computable lower bound than the maximum degree Δ for the edge chromatic number χ' because every hypergraph H satisfies $\Delta(H)\leqslant\omega(H)\leqslant\chi'(H)$. On the other hand, $\omega(H)\leqslant\chi'(H)\leqslant k(\Delta(H)-1)+1$. For $k=3$, $\Delta(H)$ and $\omega(H)$ must be even closer. Analyzing the different sorts of intersection cliques occurring, it can be shown that $\omega(H)\leqslant\frac{3}{2}\Delta(H)$, except if H consists of all 3-element subsets of some 5-element set, in which case we have $\Delta=6$ and $\omega=10$.

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